Equal temperament vs. pythagorean tuning: a mathematical plot line

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Abstract

This paper presents a formal mathematical analysis of the structural logic underlying musical scales, focusing on the theoretical tension between Pythagorean tuning and Equal Temperament. Moving beyond a purely historical survey, the study adopts a mathematico-deductive framework to demonstrate how musical pitch systems emerge from fundamental arithmetic principles and the recursive generation of intervals.The analysis first axiomatizes the Equal Tempered system as a geometric partition of the frequency spectrum, exploring its algebraic properties and transpositional invariance. Subsequently, the Pythagorean system is formalized through the powers of the $3:2$ ratio, highlighting the inherent conflict between rational purity and the necessity of a closed harmonic circle.A central contribution of this research is the systematic exploration of the number 12 as a "fixed point" in music theory. Utilizing the theory of continued fractions and Diophantine approximations, we provide a formal proof of why $N = 12$ represents a unique multidimensional optimum. We discuss the "bracketing" property, where equal-tempered degrees act as a bridge between opposing Pythagorean "shadows" - a phenomenon that remains cognitively manageable at $N = 12$ but leads to "harmonic blurring" in higher-resolution systems like $N = 53$. Ultimately, the study frames the 12-tone system not merely as a historical convenience, but as a structural necessity for the evolution of Western polyphony.

1 Introduction

The relationship between mathematics and music is as ancient as it is fascinating. This reciprocal contribution creates a unique synergy: while music provides "color" to mathematical abstractions, mathematics offers structural support to the most elusive of the arts. Although many arguments regarding this connection have been proposed - some profound, others tenuous - one fact remains certain: the scales of every musical culture are fundamentally grounded in arithmetic.
Taking a deeper perspective, we must determine whether these "musical numbers" emerge from a coherent underlying algorithm or if they constitute merely an arbitrary list without internal mechanisms. If a scale is to be more than a collection of frequencies, it must possess an *intrinsic structure*.
We will discuss mathematical algorithms designed to generate a musically coherent harmonic system. We will examine how specific recursive processes define the internal geometry of a scale, moving from a simple inventory of pitches to a rigorous study of structural symmetry.
While the core subjects addressed here are well-established and part of a long historical tradition, this study places significant emphasis on their formal presentation. We believe that an innovative perspective on the two most prominent sound systems—the Pythagorean and the Equal Tempered—can reveal deeper insights into their internal logic.

1.1 A brief historical overview

The history of music theory is characterized by a vast and heterogeneous landscape of tuning systems, reflecting a diverse array of cultural, geographical, and aesthetic priorities. From the complex microtonal divisions of Arabic maqam[1] to the shifting temperaments of the European Renaissance, the search for a

[1] Maqam refers to the system of melodic modes used in traditional Arabic music.

definitive organization of pitch has produced countless solutions. However, in the present study, we focus on the two systems that most transparently emerge from mathematical first principles: Pythagorean tuning and Equal Temperament. These two models represent the polarities of musical thought - rational purity versus geometric symmetry - and provide the essential theoretical keys to framing the broader evolution of musical scales.2.1

1.1.1 Pythagorean tuning: The Harmony of the Tetraktys

Our mathematical framing of the "Octave Principle" finds its historical root in the Pythagorean tetraktys - the arithmetic progression of the first four natural numbers $(1, 2, 3, 4)$. In the Pythagorean tradition, these numbers represented the mathematical perfection of the cosmos. When applied to acoustics, the ratios between them define the "perfect" consonances: the $2 : 1$ (diapason), the $3 : 2$ (diapente), and the $4 : 3$ (diatessaron).
The term "fifth" derives from the ancient Greek dia pente (literally "through five"), denoting the interval $I(f_0, \frac{3}{2}f_0)$. This name reflects the internal structure of the diatonic scale: the interval from C to G, for instance, encompasses five notes (C-D-E-F-G). For millennia, following the legacy of Boethius's De Institutione Musica (c. 510 AD) [2], Western theory maintained that only these simple integer ratios reflected the divine order of the "Music of the Spheres."

1.1.2 Equal Temperament: From Rational Purity to Irrational Utility

The transition from the rational ratios of the Greeks to the formal mathematical definition of Equal Temperament represents one of the most significant paradigm shifts in musicology. While various "well-tempered" approximations existed in practice to mitigate the acoustic discrepancies of keyboard instruments (as documented by Gioseffo Zarlino in his *Le Istitutioni Harmoniche*, 1558 [10]), the exact formulation using irrational numbers emerged only in the late 16^{th} century.
Prince Zhu Zaiyu in China (1584) provided the first precise calculation of the 12th root of 2 ($\sqrt[12]{2}$), a feat mirrored in Europe by the mathematician Simon Stevin [9] (Vande Spiegheling der Singconst, c. 1585) and later formalized by Marin Mersenne in his monumental Harmonie Universelle (1636) [6].
The centuries-long delay in adopting this model was due to two formidable barriers:

- the computational hurdle: Before John Napier's invention of logarithms in 1614, calculating $\sqrt[12]{2}$ with high precision was an arduous task; without the logarithmic shortcut, extracting successive roots required complex manual algorithms, making a rigorous numerical definition nearly impossible for instrument makers.
- The philosophical barrier: In the Pythagorean worldview, an irrational number was alogos (unreasonable). To accept $\sqrt[12]{2}$ meant deliberately "tempering" (distorting) the divine $3 : 2$ ratio to close the circle of fifths.The eventual triumph of Equal Temperament marked the shift from a cosmological view of music to one of practical utility, granting the modulatory freedom that defines modern Western composition.

1.2 Outline of the Work

The present study is organized as follows. We begin in Section 2 by establishing the foundational musical concepts necessary for a rigorous understanding of the subsequent analysis. Distinguishing our approach from a purely historical survey, we adopt a *mathematico-deductive framework* to demonstrate that the internal logic of musical scales stems directly from the Octave Principle. This principle clarifies why the appropriate metric for musical distance is the ratio rather than linear difference.
In Section 3, we shift to the mathematical modeling of pitch systems. Diverging from the chronological order of history, we first present Equal Temperament, as it offers the most transparent and systematic axiomatic structure. Subsequently, we introduce the mathematics underlying the Pythagorean system, which is profoundly compelling from a conceptual standpoint yet presents significant practical challenges in its concrete musical implementation. For both systems, we provide a musical interpretation of their characterizing mathematical properties.
Section 4 advances the central thesis of this work: the unique status of the number 12. While the optimality of a 12-tone division is a well-established topic, we offer fresh procedural insights into why 12 represents the ideal compromise between harmonic theory and musical practice.

Finally, Section 5 provides concluding remarks and addresses a more general problem concerning the interaction between the Equal and Pythagorean systems, specifically regarding the *bracketing* of Pythagorean pairs around an equal-tempered pitch.

2 Key musical concepts

The present section provides a concise presentation of the key musical principles and terminology that form the basis of our analysis.

2.1 The principle of octave

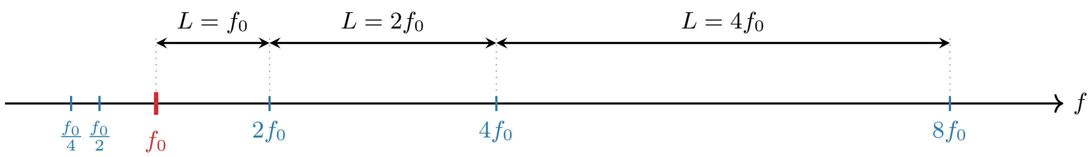
At the heart of Jean-Philippe Rameau's Traité de l'harmonie [7] lies the principle of the identity of octaves (identité des octaves). Rameau posits that the octave is a "replica[2]" of the fundamental unit, asserting that any pitch with a frequency f is essentially identical in nature to its multiples $2^k f$ (where $k \in \mathbb{Z}$)[3].
For Rameau, these sounds are not merely similar; they are the same sonorous entity manifested across different registers. The ear perceives them as nearly indistinguishable, and a pitch outside one's vocal or audible range is instinctively "replaced" by its octave equivalent.
This Principle is supported by two primary arguments:

- Mathematical Simplicity: halving a vibrating string or pipe - the simplest possible physical division - doubles its frequency[4]. This reflects the Pythagorean view of a universe regulated by Number, where music is an exact science and a simple $1:2$ ratio inevitably yields a "perfect" musical effect.
- Physical Resonance: the first harmonic of any complex tone is its double frequency. The vibrational mode with a node at the string's midpoint - vibrating at $2f$ - is the first permitted mode after the fundamental. Thus, the octave is naturally embedded within the fundamental sound itself.

The universality of this Principle is evidenced by the convention of assigning the same name to equivalent pitches. For instance, a piano features eight "A" notes ranging from 27.5 Hz to 3520 Hz, all perceived as functionally identical. The term octave derives from the fact that, in the diatonic succession of seven sounds- C, D, E, F, G, A, B - the starting pitch recurs as the eighth note (C). This eighth position, often denoted as [C] to signify the completion of the cycle, possesses exactly double the frequency of the initial tone. This concept is a cross-cultural musical universal; even in Ancient Greece, the interval was termed dia pason ("through all"), signifying the completion of a cycle before encountering the starting pitch's identity.

2.2 Distance on the Scale: Intervals

To establish a formal scale, we conceptualize the frequency spectrum as a continuous line. Given a reference frequency f_0, we identify the sounds $2^k f_0$ (where $k \in \mathbb{Z}$) as its octaves above and below.



As shown above, frequency intervals perceived as "equal" from a musical standpoint do not correspond to equal Euclidean lengths L. The linear distance between $2^k f_0$ and $2^{k+1} f_0$ is $2^k f_0$, which doubles at every

[2]In his *Traité*, Rameau frequently uses the term *réplique* (replica) to designate the octave and its multiples, emphasizing that these intervals do not introduce new harmonic content, but merely "repeat" the fundamental unit in different registers.
[3]While Rameau framed his observations through the physical lengths of a vibrating string ($1:2$ ratio), his logic aligns with the modern concept of frequency.
[4]According to Mersenne's Law, frequency is inversely proportional to length, assuming constant tension and mass.

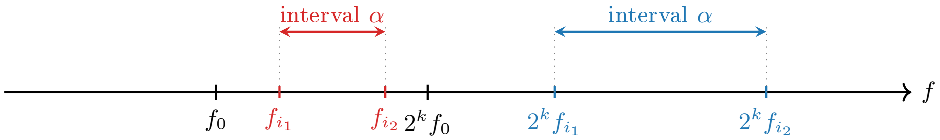
octave. To treat these segments as equivalent, we must evaluate the **ratio** between boundary frequencies rather than their linear difference:

$$\frac{2^{k+1} f_0}{2^k f_0} = 2 \quad \forall k \in \mathbb{Z}.$$

This principle extends to all intermediate frequencies. If f_{i_1} and f_{i_2} are two frequencies within the reference octave $[f_0, 2f_0]$, their counterparts in any other octave $[2^k f_0, 2^{k+1} f_0]$ will be $2^k f_{i_1}$ and $2^k f_{i_2}$. Since these pairs fulfill the same functional role within their respective scales, they must represent the same "distance." This is realized by measuring the ratio, as:

$$\frac{2^k f_{i_1}}{2^k f_{i_2}} = \frac{f_{i_1}}{f_{i_2}} \quad \text{for any } k \in \mathbb{Z}.$$

Graphically, the validity of the Principle of the Octave implies the equality of the segments highlighted below:



In the frequency domain, the significant metric for measuring distance is the ratio. We define the **musical interval** I between two frequencies f_{i_1} and f_{i_2} (where $f_{i_2} \geq f_{i_1}$) as:

$$I(f_{i_1}, f_{i_2}) = \frac{f_{i_2}}{f_{i_1}}. \tag{1}$$

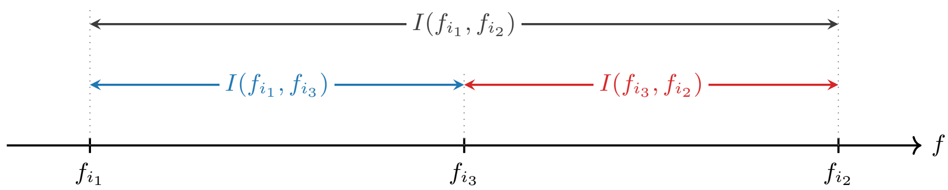
This definition ensures that $I(2^k f_{i_1}, 2^k f_{i_2}) = I(f_{i_1}, f_{i_2})$, preserving the identity of intervals across different registers. Common examples include the unison ($I = 1$) and the octave ($I = 2$).

Remark 1 *The recognition of a musical melody depends strictly on these ratios rather than absolute frequencies. Thus, the interval possesses greater structural and cognitive significance than the absolute pitch.*

Defining the interval as a ratio (1) leads to a fundamental property of interval composition: the "sum" of two adjacent musical intervals corresponds to the mathematical product of their ratios. If an interval $I(f_{i_1}, f_{i_2})$ includes an intermediate frequency f_{i_3} (where $f_{i_1} \leq f_{i_3} \leq f_{i_2}$), then:

$$I(f_{i_1}, f_{i_2}) = I(f_{i_1}, f_{i_3}) \cdot I(f_{i_3}, f_{i_2}) = \frac{f_{i_3}}{f_{i_1}} \cdot \frac{f_{i_2}}{f_{i_3}} = \frac{f_{i_2}}{f_{i_1}}. \tag{2}$$

When representing intervals graphically, it becomes clear that a segment formed by the juxtaposition of two adjacent musical distances has a total length equal to the product of its components:



This multiplicative property remains consistent even across multiple octaves. For example, combining an interval of two octaves (2^2) with one of three octaves (2^3) yields a total distance of five octaves:

$$I(f_0, 4f_0) \cdot I(4f_0, 32f_0) = 4 \cdot 8 = 32 = 2^5.$$

2.3 The Pitch System as Steps on a Scale

Within the framework of a scale $[f_0, 2f_0]$, any interval I is bounded by the unison and the octave $(1 \leq I \leq 2)$. Consequently, the tuning system of the scale can be fully defined in two equivalent ways:

- By the N **ratios** between consecutive frequencies:

$$G_1 = \frac{f_1}{f_0}, \quad G_2 = \frac{f_2}{f_1}, \quad \ldots, \quad G_N = \frac{2f_0}{f_{N-1}} \tag{3}$$

  which represent the *steps* of the scale.

- By the $N-1$ **intervals** that each sound forms with the fundamental f_0:

$$S_1 = \frac{f_1}{f_0}, \quad S_2 = \frac{f_2}{f_0}, \quad \ldots, \quad S_{N-1} = \frac{f_{N-1}}{f_0} \tag{4}$$

  which represent the *scale degrees* or *relative intervals*.

Given f_0 and the octave $f_N = 2f_0$, only $N-1$ intermediate values $f_1, \ldots, f_{N-1}$ are required to derive both the steps G_n and the degrees S_n. This mathematical structure ensures that any transposed scale $[2^k f_0, 2^{k+1} f_0]$ is partitioned by identical ratios, maintaining an invariant internal geometry across all octaves for any $k \in \mathbb{Z}$.

2.4 Defining the tuning system

Based on the common foundation of the Octave Principle, the formation of a specific scale depends on the choice of N (the number of notes) and the definition of the steps G_n (3). These selections are generally guided by two distinct criteria:

(A) Cultural and Structural Context (N): The value of N is typically established by a musical culture or genre. Western music has predominantly standardized $N = 12$, meaning that after twelve steps from a starting frequency f_0, one reaches the octave $2f_0$. This is physically evident in the twelve keys (including chromatic tones) required to span an octave on a piano or the twelve frets on a guitar. While $N = 12$ defines the *chromatic scale*, other systems are equally viable: $N = 5$ produces the pentatonic scale, $N = 6$ the whole-tone scale, and $N = 7$ the diatonic and various modal or non-Western scales.

(B) Tuning and Mathematical Constraints (G_n): Within a chosen N, fixing the steps $G_1, \ldots, G_N$ constitutes the act of *tuning*. Historically, this has moved between two paradigms: an arithmetic approach, where intermediate frequencies are derived from simple integer ratios based on f_0, and a practical approach designed to resolve the mechanical and harmonic limitations of fixed-pitch instruments.

The choice of N is a cultural phenomenon; there are no spontaneous mathematical reasons that dictate a universal number of sounds. However, point (B)—the determination of specific pitch values—is intrinsically linked to mathematics. Historically, addressing this issue in purely arithmetic terms essentially involves a choice between rational and irrational numbers.

3 The Role of Mathematics

This section examines two significant scale constructions through a mathematical lens, highlighting the specific axioms and properties that define each system beyond the fundamental Octave Principle. In an overall view, the mathematical framework underlying the two musical systems, Equal temperament and Pythagorean Tuning, can be outlined as follows:

- Octave principle + equal scale partitioning: Equal temperament,
- Octave principle + inclusion of $3^n f_0$ pitches: Pythagorean tuning.

3.1 Equal tempered scale

From a mathematical perspective, the simplest way to establish a scale is to arrange the frequencies f_0, f_1, ..., $f_N = 2f_0$ equidistantly (in the sense of the distance defined in (1)) within the interval $[f_0, 2f_0]$. The case $N = 1$ is meaningless, hence we set $N \geq 2$. Equidistance amounts to requiring that the steps $G_1, \ldots, G_N$ from (3) are all equal to a positive number r:

$$G_1 = G_2 = \cdots = G_N = r \quad \text{or} \quad \frac{f_1}{f_0} = \frac{f_2}{f_1} = \cdots = \frac{2f_0}{f_{N-1}} = r.$$

To calculate r, one can write, for example $2 = \frac{2f_0}{f_0} = \frac{2f_0}{f_{N-1}} \frac{f_{N-1}}{f_{N-2}} \cdots \frac{f_2}{f_1} \frac{f_1}{f_0} = r^N$ from which $r = 2^{1/N}$.

In order to obtain the values (4), we use the rule (2) starting from $S_1 = \frac{f_1}{f_0} = r = 2^{1/N}$:

$$S_2 = \frac{f_2}{f_0} = \frac{f_2}{f_1}\frac{f_1}{f_0} = r^2 = 2^{2/N}, \quad S_3 = \frac{f_3}{f_0} = r^3 = 2^{3/N}, \quad \cdots \quad S_{N-1} = \frac{f_{N-1}}{f_0} = r^{N-1} = 2^{(N-1)/N}$$

and finally, as a confirmation, $S_N = \frac{2f_0}{f_0} = r^N = 2$. In a compact form, we can write

$$f_k = 2^{k/N} f_0, \qquad k = 0, 1, \ldots, N. \tag{5}$$

To obtain the complete range of equal-tempered sounds generated by f_0, according to the Octave Principle, the set (5) must be replicated across each octave by multiplying it by (positive and negative) powers of 2. Therefore, the set of equal tempered sounds generated by the fundamental frequency f_0 is defined as:

$$\mathbb{T}_N(f_0) = \Big\{ f \in \mathbb{R} \,|\, f = f_{m,k} := 2^{m+k/N} f_0,\ k \in 0, 1, \ldots, N-1,\ m \in \mathbb{Z} \Big\}. \tag{6}$$

Notice that the value $k = N$ is excluded to avoid redundancy, since $f_{m,N} = 2^{m+N/N} f_0 = 2^{m+1} f_0 = f_{m+1,0}$.

The frequency $f_{m,k}$ is thus characterized by two discrete parameters:

m: the octave index, determining the frequency range $[2^m f_0, 2^{m+1} f_0)$;

k: the step index, representing the number of equal-tempered intervals ($2^{1/N}$) added to the m-th octave.

3.1.1 Structural properties of $\mathbb{T}_N(f_0)$

We emphasize the following mathematical properties of the set $\mathbb{T}_N(f_0)$.

- **Uniqueness**: each frequency $f_{m,k} \in \mathbb{T}_N(f_0)$ is uniquely determined by the pair of indices (m, k), where $m \in \mathbb{Z}$ and $k \in \{0, 1, \ldots, N-1\}$.

  Proof: suppose $f_{m_1,k_1} = f_{m_2,k_2}$. This implies $2^{m_1+k_1/N} = 2^{m_2+k_2/N}$, which leads to:

  $$m_1 - m_2 = \frac{k_2 - k_1}{N}$$

  The left-hand side is an integer, so the right-hand side must also be an integer. However, since both k_1 and k_2 are in the range $[0, N-1]$, their difference satisfies $|k_2 - k_1| < N$. The only integer in the interval $(-1, 1)$ is zero; thus, we must have $k_2 - k_1 = 0$, which implies $k_1 = k_2$. Consequently, $m_1 - m_2 = 0$, leading to $m_1 = m_2$.

- **Irrationality**: the normalized frequencies $f_{m,k}/f_0 = 2^{m+k/N}$, with $k \in \{1, \ldots, N-1\}$ and $m \in \mathbb{Z}$, are all irrational for any integer $N \geq 2$.

  Proof: Assume $2^{m+k/N} = p/q$ for some $p, q \in \mathbb{Z}^+$. Elevating both sides to the power of N, we obtain $2^{Nm+k} = p^N/q^N$, which implies:

  $$q^N \cdot 2^{Nm+k} = p^N$$

By the Fundamental Theorem of Arithmetic, the exponent of the prime factor 2 must be a multiple of N on both sides. Let $v_2(n)$ be the 2-adic valuation (the exponent of the highest power of 2 in the prime factorization of n). On the right side, the exponent is $v_2(p^N) = N \cdot v_2(p)$, which is a multiple of N. On the left side, the exponent is:

$$v_2(q^N \cdot 2^{Nm+k}) = N \cdot v_2(q) + Nm + k$$

Since $1 \leq k < N$, this sum cannot be a multiple of N, leading to a contradiction. $\square$

- **Discreteness**: the set (6) is a discrete subset of $\mathbb{R}^+$. A subset $S \subseteq \mathbb{R}$ is discrete if every point $s \in S$ is an isolated point; that is, there exists a neighborhood U of s such that $U \cap S = \{s\}$. In our case, for any $f \in \mathbb{T}_N(f_0)$, the distance to the nearest neighbor is at least:

  $$\Delta f \geq f\left(2^{1/N} - 1\right) > 0$$

  This ensures that no point of the set is an accumulation point.

- **Geometric sequence**: when ordered, the elements of $\mathbb{T}_N(f_0)$ constitute a geometric progression with common ratio $q = 2^{1/N}$. The natural ordering consists of arranging the elements $f_{m,k}$ lexicographically by the index pair (m, k). Specifically, for any two frequencies f_{m_1,k_1} and f_{m_2,k_2}, we set $f_{m_1,k_1} < f_{m_2,k_2}$ if $m_1 < m_2$, or $m_1 = m_2$ and $k_1 < k_2$. Under this ordering, each element is obtained by multiplying the preceding one by $2^{1/N}$, effectively mapping the discrete steps of the scale to a continuous exponential growth.

- **Partition of $\mathbb{R}^+$ into Equal-Tempered Sets**: the following properties

  1. $f_1 \in \mathbb{T}_N(f_2) \iff f_2 \in \mathbb{T}_N(f_1)$,
  2. If $\mathbb{T}_N(f_1) \cap \mathbb{T}_N(f_2) \neq \emptyset$, then $\mathbb{T}_N(f_1) = \mathbb{T}_N(f_2)$

  entail that the "membership in the same equal-tempered set" relation constitutes an equivalence relation over $\mathbb{R}^+$. In algebraic terms, each set $\mathbb{T}_N(f_0)$ is an equivalence class of the multiplicative quotient group $\mathbb{R}^+/\langle 2^{1/N} \rangle$, where $\langle 2^{1/N} \rangle$ is the cyclic subgroup generated by the N-th root of 2, defined as:

  $$\langle 2^{1/N} \rangle = \{2^{z/N} : z \in \mathbb{Z}\}.$$

  This subgroup represents the set of all possible intervals (expressed as frequency ratios) that can be formed using the fundamental step of the N-tone equal temperament.

- **Transpositional Invariance**: consider an arbitrary subset $S \subset \mathbb{T}_N(f_0)$ and, for a fixed integer $\tau \in \mathbb{Z}$, define the transposed set

  $$S_\tau = \{r^\tau f \mid f \in S\} \tag{7}$$

  generated by shifting each element of the original set forward ($\tau > 0$) or backward ($\tau < 0$) by $|\tau|$ steps of the common ratio $r = 2^{1/N}$. Then, the musical interval between any two elements $f_1, f_2 \in S$ is preserved for the corresponding elements in S_τ:

  $$I(r^\tau f_1, r^\tau f_2) = \frac{r^\tau f_2}{r^\tau f_1} = \frac{f_2}{f_1} = I(f_1, f_2).$$

Remark 2 *From an algebraic perspective, the transpositional invariance is a direct consequence of the multiplicative group structure of* $(\mathbb{R}^+, \cdot)$. *Since* $\mathbb{T}_N(f_0)$ *is a coset of the cyclic subgroup* $\langle r \rangle$, *the transposition map* $T_\tau : f \mapsto r^\tau f$ *is an automorphism (specifically, a translation in the group) that preserves the ratio between any two elements.*

3.1.2 Musical counterparts: the tempered grid

We examine the properties just listed from an acoustic-musical perspective.

- *Irrationality*: historically, the equal scale was criticized for using "unnatural" irrational numbers. Indeed, they do not align with the harmonic series or simple integer ratios traditionally used for obtaining consonant sounds. In plain terms, fitting N equal segments into the interval $[1, 2]$ — where distance is defined as in (1) — provides a manageable finite set of sounds. However, these cannot be obtained through simple practical operations, such as integer string divisions, as they inherently involve irrational proportions.

- *Geometric sequence*: the psychological foundation of the equal-tempered scale is rooted in its geometric progression. To achieve a uniform increase in perception, stimuli must follow a geometric progression, in accordance with Weber's Law of psychophysics. Although defined by irrational numbers, these intervals satisfy the human need for a linear, uniform progression in perceived pitch, mapping the logarithmic nature of our hearing onto an arithmetic-like grid.

- *Discreteness*: this property is the mathematical prerequisite for the existence of distinct "notes" and musical notation. By ensuring that each element is an isolated point, the system provides the perceptual stability necessary for a musical language. The distance Δf guarantees that each pitch remains identifiable and distinct from its neighbors, preventing the sonic space from collapsing into a continuous, undifferentiated slide (glissando).

- *Partition of* $\mathbb{R}^+$ *into Equal-Tempered Sets*: the "space of all possible sounds" is divided into infinitely many parallel equal-tempered systems. By choosing a reference frequency (e.g., $A4 = 440$ Hz), we define the standard tempered system used in Western music. Any frequency not belonging to $\mathbb{T}_{12}(440)$, such as 441 Hz, belongs to a different block of the partition — effectively a system that is either "out of tune" or shifted relative to the standard one.

- *Transpositional Invariance*: this property invites a fundamental observation. Let S be a finite set $S\{f_{i_1}, f_{i_2}, \dots, f_{i_\kappa}\}$: these sounds can represent a chord (if simultaneous), a melodic line (if sequential), a scale (if arranged in ascending order and governed by a specific rule), or the whole scale itself, if the increment is the constant interval r.

Example 1 *In the twelve-tone equal temperament system ($N = 12$), the following sets represent standard musical structures within a generic octave $m \in \mathbb{Z}$:*

$\{f_{m,0}, f_{m,1}, \dots, f_{m,11}, f_{m+1,0}\}$	*Chromatic scale*
$\{f_{m,0}, f_{m,2}, f_{m,4}, f_{m,5}, f_{m,7}, f_{m,9}, f_{m,11}, f_{m+1,0}\}$	*Major diatonic scale*
$\{f_{m,0}, f_{m,2}, f_{m,3}, f_{m,5}, f_{m,7}, f_{m,8}, f_{m,10}, f_{m+1,0}\}$	*Natural minor scale*
$\{f_{m,0}, f_{m,4}, f_{m,7}\}$, $\{f_{m,0}, f_{m,3}, f_{m,7}\}$	*Major chord, minor chord*
$\{f_{m,0}, f_{m,4}, f_{m,7}, f_{m,11}\}$, $\{f_{m,0}, f_{m,3}, f_{m,6}, f_{m,9}\}$	*Major 7th chord, diminished 7th chord.*

Moreover, the first theme of Brahms's Fourth Symphony can be described as a sequence of frequencies across octaves:

$$\{f_{m,11}, f_{m,7}, f_{m,4}, f_{m+1,0}, f_{m,9}, f_{m,6}, f_{m,3}, f_{m,11}\}$$

The use of the generic index m emphasizes that these musical structures are defined solely by the relative differences between the k indices, regardless of the absolute frequency range (the register) in which they are performed.

Each set of sounds is completely defined by the number of constant steps r between one frequency and the next; the transposition (7) applies the same succession of intervals starting from a different f_0. Therefore, in the equal temperament system, up to a translation τ, there exists only one chromatic scale, only one major scale (leading to the common observation that "only one key exists"), only one major chord, and so forth.

The homogeneity of the equal-tempered scale ensures that scales and chords, when transposed to another key, do not change their distinctive internal relations. Similarly, from a melodic standpoint, a transposed melody traverses intervals of the same length; its structure remains unaltered, shifting only in pitch. A melody sounds identical regardless of the starting key.

In musical terms, this ensures that the internal structure of any melody or chord is invariant under shifting, making all N keys within the equal-tempered system functionally equivalent. The composer can therefore write in any key without facing constraints specific to any of them, and the performer can transpose sounds, melodies, and harmonies into the desired keys using simple arithmetic rules.

3.2 Pythagorean tuning

The atmosphere is now completely different: the structural practicality of equidistantly spacing the scale elements (in terms of ratio) is replaced by a Principle that essentially extends that of the octave. We embrace the presence, within the scale to be constructed, of the sound obtained by dividing the string into three equal parts or tripling its length. Starting from the fundamental frequency f_0, we can employ the frequencies $3f_0, 9f_0, 27f_0, \dots$ and $\frac{1}{3}f_0, \frac{1}{9}f_0, \frac{1}{27}f_0, \dots$ or, more concisely:

$$3^k f_0, \quad k \in \mathbb{Z} \text{ integers.} \tag{8}$$

Equation (8) may be referred to as the "Principle of the fifth" (see Introduction).
From the simultaneous validity of the octave and fifth principles, we deduce that the sound universe available from a fundamental frequency f_0 is

$$\mathbb{P}(f_0) = \left\{ f \in \mathbb{R} \,|\, f = p_{a,b} = 2^a 3^b f_0,\ a, b \in \mathbb{Z} \right\}. \tag{9}$$

3.2.1 Structural properties of $\mathbb{P}(f_0)$

As with the equal-tempered set, we outline below the fundamental properties of $\mathbb{P}(f_0)$.

- **Uniqueness**: each frequency $p_{a,b} \in \mathbb{P}(f_0)$ is uniquely determined by the pair of integers (a, b).

  Proof: suppose there exist two pairs (a_1, b_1) and (a_2, b_2) such that $p_{a_1,b_1} = p_{a_2,b_2}$. This implies:

  $$2^{a_1} 3^{b_1} f_0 = 2^{a_2} 3^{b_2} f_0 \implies 2^{a_1 - a_2} = 3^{b_2 - b_1}.$$

  By the Fundamental Theorem of Arithmetic, since 2 and 3 are distinct prime numbers, the only way for a power of 2 to equal a power of 3 is if both exponents are zero. Therefore, $a_1 - a_2 = 0$ and $b_2 - b_1 = 0$, which means $a_1 = a_2$ and $b_1 = b_2$. $\square$

- **Rationality**: the normalized frequencies $p_{a,b}/f_0 = 2^a 3^b$, with $a, b \in \mathbb{Z}$, are all rational numbers. This property is an immediate consequence of the fact that the set $\mathbb{P}(f_0)$ is generated by integer bases. Since any integer power of 2 or 3 is an integer (for $a, b \geq 0$) or a unit fraction (for $a, b < 0$), their product necessarily belongs to the field of rational numbers $\mathbb{Q}$.

- **Density**: The set $\mathbb{P}(f_0)$ is a dense subset of $\mathbb{R}^+$. This means that for any two frequencies $f_1, f_2 \in \mathbb{R}^+$ with $f_1 < f_2$, there exists at least one Pythagorean frequency $p_{a,b}$ such that $f_1 < p_{a,b} < f_2$.

  Proof (sketch): Taking the base-2 logarithm, the condition $2^a 3^b \in (f_1/f_0, f_2/f_0)$ is equivalent to:

  $$\log_2(f_1/f_0) < a + b \log_2(3) < \log_2(f_2/f_0)$$

  Since $\log_2(3)$ is an irrational number, the set of linear combinations $\{a + b \log_2(3) : a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ (by Kronecker's Theorem). Therefore, the original set of frequencies is dense in $\mathbb{R}^+$. $\square$

- **Partition of $\mathbb{R}^+$ into Equal-Tempered Sets**: the same logic applies to the Pythagorean system. Specifically:

  1. $f_1 \in \mathbb{P}(f_2) \iff f_2 \in \mathbb{P}(f_1)$,
  2. if two Pythagorean sets share a common frequency, they are identical: $\mathbb{P}(f_1) \cap \mathbb{P}(f_2) \neq \emptyset \implies \mathbb{P}(f_1) \equiv \mathbb{P}(f_2)$.

  Consequently, the relation "belonging to the same Pythagorean set" is an equivalence relation on $\mathbb{R}^+$. In algebraic terms, $\mathbb{P}(f_0)$ is an equivalence class of the quotient group:

  $$\mathbb{R}^+ / \langle 2, 3 \rangle$$

where $\langle 2,3\rangle$ is the multiplicative subgroup of $\mathbb{R}^+$ generated by the primes 2 and 3:

$$\langle 2,3\rangle = \{2^a 3^b : a,b \in \mathbb{Z}\}.$$

The family of all Pythagorean sets constitutes a partition of the frequency space $\mathbb{R}^+$.

- **Non-closure**: the set $\mathbb{P}(f_0)$ does not contain any frequency $p_{a,b}$ that is an exact octave of f_0, except for the trivial cases where $b = 0$. This is a direct consequence of the Uniqueness property (or the Fundamental Theorem of Arithmetic). Since $2^a 3^b = 2^k$ implies $3^b = 2^{k-a}$, the only integer solution is $b = 0$.

- **Transpositional invariance**: consider an arbitrary subset $S \subset \mathbb{P}(f_0)$ and, for fixed integers $\alpha, \beta \in \mathbb{Z}$, the transposed set:
$$S_{\alpha,\beta} = \left\{(2^\alpha 3^\beta)f \mid f \in S\right\} \tag{10}$$
generated by shifting each element of the original set by α octaves and β perfect fifths. Then, the distance (the interval ratio) between any two elements $f_1, f_2 \in S$ is preserved for the corresponding elements in $S_{\alpha,\beta}$:
$$I(2^\alpha 3^\beta f_1, 2^\alpha 3^\beta f_2) = \frac{2^\alpha 3^\beta f_2}{2^\alpha 3^\beta f_1} = \frac{f_2}{f_1} = I(f_1, f_2).$$

Remark 3 *Just as in the equal-tempered case, this property stems from the fact that $\mathbb{P}(f_0)$ is a coset of the multiplicative subgroup $\langle 2,3\rangle$ within $(\mathbb{R}^+,\cdot)$. Any transposition $T_{\alpha,\beta} : f \mapsto (2^\alpha 3^\beta)f$ is an automorphism of the group structure that preserves the relative ratios between elements.*

3.2.2 Musical counterparts: the harmonic wilderness

The structural properties of the Pythagorean set $\mathbb{P}(f_0)$ lead to a musical landscape that is fundamentally different from the equal-tempered one, characterized by "pure" acoustic beauty but also by significant practical limitations. Broadly speaking, while the density of $\mathbb{P}(f_0)$ offers infinite theoretical possibilities, it creates insurmountable practical hurdles.

- *Rationality and Natural Consonance*: unlike the irrational steps of equal temperament, the Pythagorean system is built on the ratio 3/2, which corresponds to the third harmonic of the natural harmonic series[5]. The simplicity of this integer ratio reflects the ancient Pythagorean ideal that harmony is rooted in the relationship between whole numbers. The perceived consonance between f_0 and its multiples follows the simplicity of the ratio: the octave $(2:1)$ and the fifth $(3:2)$ represent the highest degrees of harmonic stability. This system ensures that perfect fifths are "pure" (beat-less).

- *Density and the Infinite Scale*: The density of $\mathbb{P}(f_0)$ in $\mathbb{R}^+$ implies that the Pythagorean system is, in principle, an infinite "open" system: musically, density implies that by continuing the cycle of fifths indefinitely, one can approximate any desired pitch with arbitrary precision. This means that a Pythagorean instrument would theoretically require an infinite number of strings or keys to maintain perfect intonation across all transpositions.

- *Non-closure and the Pythagorean Comma*: the mathematical fact that $2^a 3^b \neq 2^k$ (for $b \neq 0$) has a dramatic musical consequence: the circle of fifths does not close. After 12 fifths, instead of returning to the starting note, we reach a frequency slightly higher than the seventh octave:
$$\frac{(3/2)^{12}}{2^7} = \frac{3^{12}}{2^{19}} \approx 1.0136.$$
This discrepancy, known as the Pythagorean comma, is the "error" that all temperament systems attempt to resolve. In a pure Pythagorean system, if one insists on closing the circle with only 12 notes, the final fifth[6] will be so out of tune as to be musically unusable.

[5]The **harmonic series** is the sequence of sounds whose frequencies are integer multiples of a fundamental frequency f. In acoustical physics, when a string vibrates, it does not only produce the root note, but simultaneously oscillates in fractions of its own length $(1/2, 1/3, 1/4\ldots)$, generating what are known as *overtones*. The first terms define the "pure" intervals of the Pythagorean system: the octave $(2:1)$, the perfect fifth $(3:2)$, and the pure major third $(5:4)$.

[6]the "wolf fifth"

- *Transpositional Invariance vs. Practical Modulation*: while the infinite set $\mathbb{P}(f_0)$ is technically invariant under transposition by any interval $r = 2^\alpha 3^\beta$, a significant conflict arises in finite applications.

  In the equal-tempered system, all keys are functionally equivalent (congruent) because the octave is divided into equal logarithmic steps. In contrast, finite Pythagorean scales are not congruent. If we select a finite subset of N tones $\mathcal{S}_{f_0} \subset \mathbb{P}(f_0)$ and attempt to transpose it to a new base frequency $\tilde{f}_0 \in \mathbb{P}(f_0)$, the resulting set $\mathcal{S}_{\tilde{f}_0}$ will inevitably contain pitches outside the original selection.

  Due to the uniqueness of the prime factorization in (11), choosing a new starting point $\tilde{f}_0 = 2^h 3^k f_0$ effectively "shifts" the required exponents. For example, if we generate a 12-tone scale starting from f_0 (C), and then attempt to play a melody starting from $\tilde{f}_0 = 2^{-6}3^4 f_0$ (E), a note generated 9 fifths above $\tilde{f}_0$ would require the frequency:

  $$\tilde{f}_{new} = (3/2)^9 \cdot \tilde{f}_0 = 2^{-9}3^9 \cdot (2^{-6}3^4 f_0) = 2^{-15}3^{13} f_0$$

  This frequency belongs to $\mathbb{P}(f_0)$ but lies at $k = 13$, beyond the standard 12-tone truncation.
  [7]

  Consequently, the Pythagorean system forces a choice between perfect intonation in a few "home" keys or unacceptable dissonance (such as the wolf fifth) in others. This lack of congruence gives each key a unique "affect" or color—a property lost in the symmetry of equal temperament—but severely limits the freedom of modulation.

- *Partition and the Standard Pitch*: similar to the equal-tempered case, the partition property implies that once a fundamental f_0 is chosen, the entire “grid” of pure fifths and octaves is fixed. Any sound outside this grid (even by a few cents) belongs to a different Pythagorean world. However, due to the density of the set, the “boundaries” between different Pythagorean sets are perceptually much thinner than in the discrete equal-tempered case.

3.2.3 Generating Pythagorean scales: a fifth-based selection algorithm

Broadly speaking, while the density of $\mathbb{P}(f_0)$ allows for arbitrary pitch approximation, practical instrument design requires a *stopping criterion* to truncate the infinite cycle of fifths. The standard algorithm for generating a Pythagorean scale relies on stacking perfect fifths ($3^a f_0$) and applying *octave reduction* to map each tone back into the reference interval $[f_0, 2f_0]$.

In an overall view, this process of octave equivalence is achieved by dividing each frequency by the appropriate power of 2, such that $2^b \leq 3^a < 2^{b+1}$. Any Pythagorean tone $p_{a,b}$ generated from f_0 can thus be formally expressed by the following normalization:

$$p_{a,b} = \frac{3^a f_0}{2^b}, \quad \text{with} \quad b = \lfloor a \log_2(3) \rfloor, \quad a \in \mathbb{Z} \tag{11}$$

where the floor function $\lfloor \cdot \rfloor$ ensures the ratio remains within the fundamental octave. Note that for $k < 0$ (descending fifths), b also becomes negative, consistently maintaining the frequency within the set boundaries.
To bridge the gap between theoretical infinite density and practical musical application, we examine the distribution of tones as $|a|$ increases. By generating twelve fifths forward ($a = 1, \dots, 12$) and twelve backward ($a = -1, \dots, -12$), a highly ordered configuration emerges within the interval $[f_0, 2f_0]$. This selection reveals the "sharp" and "flat" versions of the chromatic scale, where pitches are separated by a limited set of micro-intervals.

To better isolate the generative mechanics, the expression for $p_{a,b}$ from (11) can be rewritten as:

$$p_{a,b} = 2^{a-b} \left(\frac{3}{2}\right)^a f_0, \quad a = \pm 1, \dots, \pm 12 \tag{12}$$

[7] Mathematically, a melody is a sequence $\{s_i\} \subset \mathbb{P}(f_0)$. Transposing it by $r \in \mathbb{P}$ maps $s_i \mapsto r \cdot s_i$. While $r \cdot s_i \in \mathbb{P}(f_0)$ always holds, the condition $r \cdot s_i \in \{f_1, \dots, f_N\}$ (the "keyboard") fails for any finite N because the subdivisions are not equally spaced.

In this formulation, $(3/2)^a$ represents the stack of pure fifths, while the term 2^{a-b} performs the necessary octave corrections to maintain the ratio within $[1, 2]$. Notably, the exponent $a - b$ always carries the opposite sign of a, acting as a compensatory shift for ascending ($a > 0$) or descending ($a < 0$) cycles.

The following table summarizes the normalized ratios for the first twelve steps in both directions:

Descending Fifths ($a < 0$)		Ascending Fifths ($a > 0$)	
a	$2^{a-b}(3/2)^a$	a	$2^{a-b}(3/2)^a$
-1	$2(3/2)^{-1} \approx 1.3333$	1	$(3/2)^1 = 1.5$
-2	$2^2(3/2)^{-2} \approx 1.7777$	2	$2^{-1}(3/2)^2 = 1.125$
-3	$2^2(3/2)^{-3} \approx 1.1851$	3	$2^{-1}(3/2)^3 \approx 1.6875$
-4	$2^3(3/2)^{-4} \approx 1.5802$	4	$2^{-2}(3/2)^4 \approx 1.2656$
-5	$2^3(3/2)^{-5} \approx 1.0534$	5	$2^{-2}(3/2)^5 \approx 1.8984$
-6	$2^4(3/2)^{-6} \approx 1.4046$	6	$2^{-3}(3/2)^6 \approx 1.4238$
-7	$2^5(3/2)^{-7} \approx 1.8728$	7	$2^{-4}(3/2)^7 \approx 1.0678$
-8	$2^5(3/2)^{-8} \approx 1.2485$	8	$2^{-4}(3/2)^8 \approx 1.6018$
-9	$2^6(3/2)^{-9} \approx 1.6647$	9	$2^{-5}(3/2)^9 \approx 1.2013$
-10	$2^6(3/2)^{-10} \approx 1.1098$	10	$2^{-5}(3/2)^{10} \approx 1.8020$
-11	$2^7(3/2)^{-11} \approx 1.4798$	11	$2^{-6}(3/2)^{11} \approx 1.3515$
-12	$2^8(3/2)^{-12} \approx 1.9730$	12	$2^{-7}(3/2)^{12} \approx 1.0136$

As illustrated in Fig. 1, mapping these tones onto the unit segment reveals a symmetrical distribution where the distances between adjacent pitches are limited to only three specific intervals (the Limma, the Apotome, and the Comma).

The diagram in Fig. 1 illustrates the following key features:

- **The Pythagorean Comma as a Natural Limit**: The values $a = 12$ and $a = -12$ identify two tones in close proximity to the endpoints 1 and 2 (f_0 and $2f_0$). This gap, $I_{\mathcal{C}} = 3^{12}/2^{19} \approx 1.0136$, is the Pythagorean comma. While $I_{\mathcal{C}}$ represents the structural obstacle to closing the circle, this extreme proximity suggests that $a = 12$ serves as a natural "stopping criterion" for the generative process.
- **Structural Symmetry**: The intervals between successive pitches in both the upper sequence (ascending fifths, $a > 0$, blue) and the lower sequence (descending fifths, $a < 0$, red) consist of only two types: the *limma* ($I_{\mathcal{L}} = 2^8/3^5 \approx 1.0534$) and the *apotome* ($I_{\mathcal{A}} = 3^7/2^{11} \approx 1.0678$). The identical distribution of these intervals reflects the underlying algebraic symmetry of the system.
- **Sound Zones and Intervals of a Comma**: The positioning of the 26 values defines 13 pairs of closely spaced points:

  $$(1, a = 12),\ (a = -5, a = 7),\ (a = -10, a = 2) \dots (a = -7, a = 5),\ (a = -12, 2)$$

  Each pair forms an interval of exactly one comma $I_{\mathcal{C}}$ (highlighted by the green segments). This effectively defines 13 “sound zones” each bounded by a “flat” version (red, from $a < 0$) and a “sharp” version (blue, from $a > 0$). This confirms the relation $I_{\mathcal{L}} \cdot I_{\mathcal{C}} = I_{\mathcal{A}}$, where a limma plus a comma yields an apotome.
- **The Complementary Rule**: Notably, the indices a associated with each comma-sized interval always satisfy $|a_{red}| + |a_{blue}| = 12$. This balance demonstrates how the system achieves a uniform and homogeneous distribution only upon completing the 12-fifth cycle, mapping the entire chromatic space with mathematical consistency.

4 Comparative Analysis: Pythagorean vs. Equal Tempered Scales

4.1 Disjointness and the Octave Intersection

The fundamental distinction between the two systems lies in the nature of their ratios: rationality for the Pythagorean set $\mathbb{P}(f_0)$ and algebraic irrationality for the Equal-Tempered set $\mathbb{T}_N(f_0)$. These two infinite worlds intersect exclusively at pure octave intervals:

$$\mathbb{P}(f_0) \cap \mathbb{T}_N(f_0) = \{f_0 \cdot 2^m : m \in \mathbb{Z}\}. \quad (13)$$

Distribution of Pythagorean Intervals $(I_{\mathcal{L}}, I_{\mathcal{A}}, I_{\mathcal{C}})$

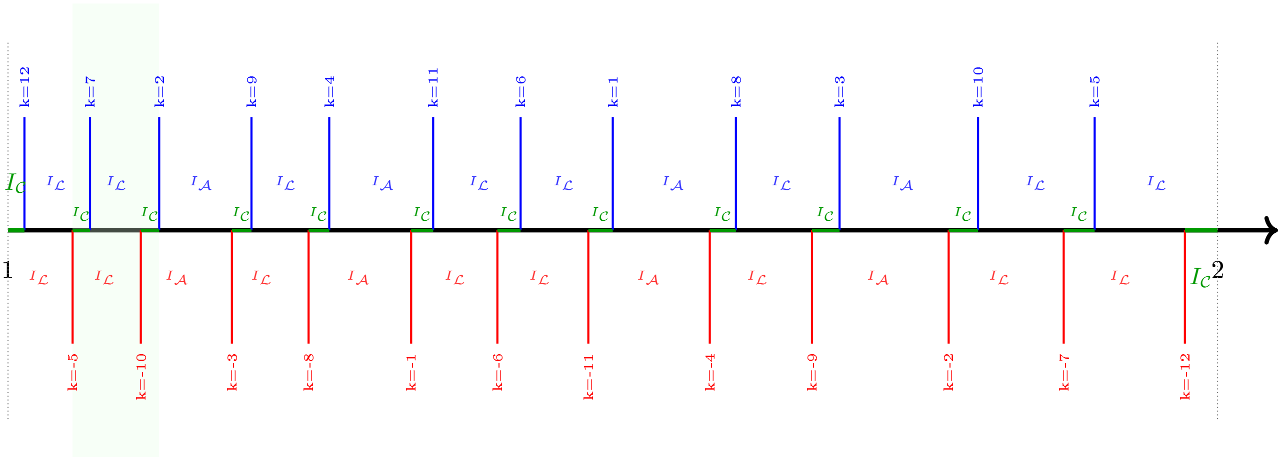


Figure 1: Mapping of the generated Pythagorean tones up to $|k| = 12$. The emergence of only three distinct interval sizes $I_{\mathcal{L}}, I_{\mathcal{A}}, I_{\mathcal{C}}$ and the appearance of 13 close-set pairs highlight the ordered nature of the system.

Proof sketch: If a frequency belonged to both sets, then $2^b 3^a = f_0 \cdot 2^{m+k/N}$. This implies $3^a = 2^{m-b+k/N}$. Raising to the N-th power yields $3^{Na} = 2^{N(m-b)+k}$. Due to the uniqueness of prime factorization, this equality holds if and only if both exponents are zero ($a = 0$ and $k = 0$), corresponding to pure octaves.

4.2 The 12-Tone Horizon: From Duality to Unity

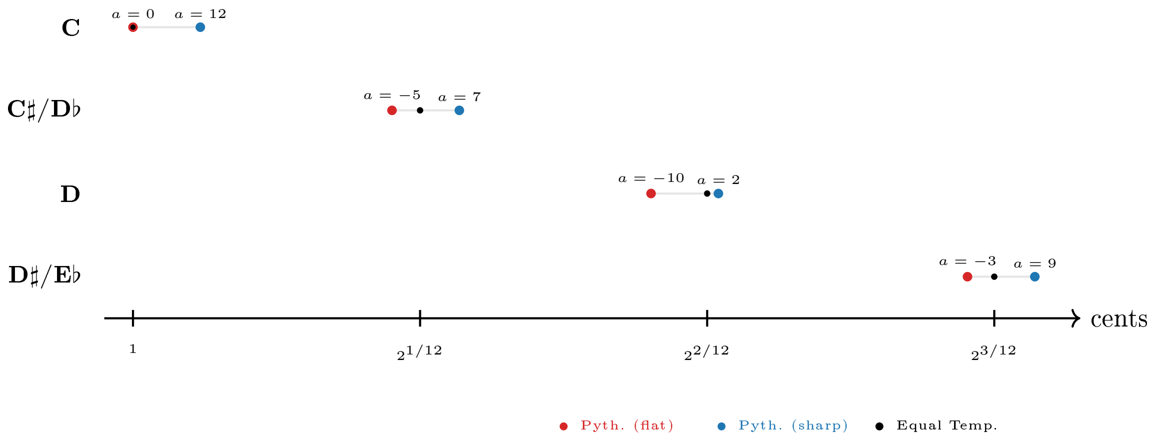
In modern practice, music employs $N = 12$ for the equal-tempered scale[8]. For $N = 12$, the presence of divisors 2, 3, and 4 predisposes the octave to various symmetries and significant combinations of sub-intervals; it would be necessary to reach $N = 60$ to encounter an additional divisor.
When we map our 26 Pythagorean tones ($a = \pm 1, \ldots, \pm 12$) against these 12 degrees, a remarkable "bracketing" emerges. Each equal-tempered degree is framed by two Pythagorean approximations: one by defect (flat) and one by excess (sharp).

Equal Tempered Degree ($\mathbb{T}_{12}$)		Pythagorean Bracketing ($p_{a,b}$)	
1.0000	C (∗)	1	$2^{-7}(3/2)^{12}$
1.0595	C♯/D♭	$2^{3}(3/2)^{-5}$	$2^{-4}(3/2)^{7}$
1.1225	D (∗)	$2^{-1}(3/2)^{2}$	$2^{6}(3/2)^{-10}$
1.1892	D♯/E♭	$2^{2}(3/2)^{-3}$	$2^{-5}(3/2)^{9}$
1.2599	E (∗)	$2^{-2}(3/2)^{4}$	$2^{5}(3/2)^{-8}$
1.3348	F (∗)	$2(3/2)^{-1}$	$2^{-6}(3/2)^{11}$
1.4142	F♯/G♭	$2^{4}(3/2)^{-6}$	$2^{-3}(3/2)^{6}$
1.4983	G (∗)	$3/2$	$2^{7}(3/2)^{-11}$
1.5874	G♯/A♭	$2^{3}(3/2)^{-4}$	$2^{-4}(3/2)^{8}$
1.6818	A (∗)	$2^{-1}(3/2)^{3}$	$2^{6}(3/2)^{-9}$
1.7818	A♯/B♭	$2^{2}(3/2)^{-2}$	$2^{-5}(3/2)^{10}$
1.8878	B (∗)	$2^{-2}(3/2)^{5}$	$2^{5}(3/2)^{-7}$
2.0000	C (∗)	2	$2^{8}(3/2)^{-12}$

For diatonic tones (∗), the table prioritizes the value reached with the lower $|a|$, reflecting the musical principle that stability correlates with harmonic proximity. A profound symmetry is observed: in every

[8] Starting from C as f_0, the twelfe steps define the chromatic scale on a piano keyboard, encompassing all white and black keys between two adjacent C octaves.

row, the sum of the absolute values of the exponents $|a|$ is exactly 12. This confirms that each degree is consistently defined by a pair of tones that together complete the twelve-step horizon.



4.3 The Resolution of the Duality

In this light, the 12-tone Equal Temperament serves as the ultimate resolution of the Pythagorean duality inherent in the 26-tone system. Our analysis reveals that the ordering within the table follows a specific harmonic logic: for diatonic tones $(*)$, the value reached through fewer iterations (i.e., the lower $|a|$ value) is listed first. This reflects a fundamental musical principle where stability and clarity are associated with a smaller number of steps within the cycle of fifths. These tones represent the core of the scale, achieved through the most direct harmonic relationships.

Specifically, for a diatonic note like D, the system yields two possibilities: one reached in only 2 steps $(a = 2)$ and another reached through 10 steps in the opposite direction $(a = -10)$. By prioritizing the $a = 2$ value, we emphasize the harmonic proximity of the diatonic tones compared to their more distant, enharmonic[9] counterparts.

Furthermore, a remarkable mathematical symmetry emerges from this arrangement: for every row in the table, the sum of the absolute values of the two exponents $|a_1| + |a_2|$ is exactly 12. For instance, for the note D $(*)$, we have $|2| + |-10| = 12$, and for G $(*)$, $|1| + |-11| = 12$. This property confirms that each equal-tempered degree is consistently "bracketed" by two Pythagorean tones that together complete a full cycle of twelve fifths, highlighting the structural balance of the 26-tone system where every pitch is defined by its relative distance—either forward or backward—within the same twelve-step horizon.

The transition to the 12-tone Equal Temperament effectively collapses this "gap" or "shadow" into a single, unified value. Mathematically, by distributing the Pythagorean comma (I_C) equally across all twelve fifths of the cycle, we force the two distant ends $(a = 12$ and $a = -12)$ to meet at the same point. This "closing" of the circle replaces the systematic bracketing with a single geometric mean.

While this process sacrifices the harmonic purity of the perfect fifths—which are narrowed by approximately 2 cents—it grants the system its most powerful property: transpositional invariance. By erasing the distinction between "flat" and "sharp" approximations, the equal-tempered system ensures that every key and every interval remains perfectly congruent, regardless of the starting pitch f_0.

In summary, this represents a fundamental shift from harmonic purity to structural symmetry. While the Pythagorean system preserves the perfect 3/2 ratio, it results in a non-congruent set of pitches where every transposition alters the internal geometry of a melody. Equal Temperament "absorbs" the comma to achieve a perfectly uniform musical space—a subtle tempering of the pure fifth that is the necessary price for a universal harmonic landscape.

[9] In modern musical notation, enharmonic refers to notes that have different names but sound at the same pitch (e.g., C♯ and D♭). In the context of this study, it refers to the distinct Pythagorean approximations that converge into a single equal-tempered degree.

5 Conclusion: The 12-Tone Horizon as a Mathematical Fixed Point

A novel contribution of this work lies in the formal presentation and systematic organization of the two musical systems, where mathematical priority serves as the guiding principle. By treating the Pythagorean and equal-tempered systems not merely as historical artifacts but as rigorous algebraic structures, we have been able to map the precise mechanics of their interaction. This formalization allows us to see that the twelve-tone system is not a mere human invention, but a fundamental discovery—a "fixed point" where the infinite spiral of Pythagorean fifths almost perfectly grazes the closed circle of the powers of two.

The analysis confirms the uniquely privileged position of the number twelve within the spectrum of possible scale divisions. While exploring denser systems, such as the fifty-three-tone scale, means zooming deeper into this numerical coincidence, attempting to impose symmetry on arbitrary divisions—such as a twenty-tone scale—is an attempt to force geometric regularity where the nature of numbers provides none [5].

The "bracketing" property we have observed—the capacity of each equal-tempered key to act as a "vice" that holds and unifies two different Pythagorean shadows—is a quality belonging only to specific nodes of resonance. These nodes, identified by the theory of continued fractions [4], ensure that the irrational value of the natural fifth is systematically sandwiched between approximations of excess and deficiency. In this sense, the twelve-tone system represents the first great "unifier" in musical history, capable of condensing the infinite density of natural resonances into a discrete and functional grid [3].

However, the comparison between the twelve and fifty-three-tone systems raises a fascinating dilemma between acoustic perfection and cognitive efficiency. Although the fifty-three-tone system is mathematically superior in capturing the purity of fifths, it falls victim to a kind of "harsh rigor." By being too faithful to its Pythagorean derivation, it inherits the historical flaw of the latter: a major third that is excessively tense and distant from the natural resonance of upper harmonics. Conversely, the twelve-tone system achieves a felicitous compromise. By slightly narrowing the fifths to close the circle, it inadvertently shifts the major thirds toward a point of greater natural consonance—a feat that the atomic precision of the fifty-three-tone system could never replicate [1].

Beyond harmonic ratios, the stability of human perception comes into play. In the twelve-tone system, intervals maintain a clear and distinct melodic identity; the steps of the scale are solid, recognizable "rungs" that allow the brain to navigate the musical discourse. In contrast, the extreme density of the fifty-three-tone system atomizes the hierarchy of intervals, transforming the scale into a nearly continuous ramp of micro-variations. In such a crowded environment, the logic of tension and resolution dissolves into a sonic blur, where the perfection of mathematical ratios is paid for with the loss of melodic intelligibility [8].

Ultimately, the twelve-tone system emerges as a multi-dimensional optimum. It balances the need for pure fifths with a tolerable approximation of major thirds, while maintaining an algebraic symmetry that allows for total freedom of modulation. Beyond this horizon, the mathematical plot line leads into a forest of intervals so dense they challenge the human capacity to categorize sound, confirming that the choice of twelve is not merely a historical convention, but a structural necessity for the evolution of Western polyphony.

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